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WHAT IS THE MAIN DIAGONAL OF A BIINFINITE BAND MATRIX?(U)
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WHAT IS THE MAIN DIAGONAL OF A BIINFINITE BAND MATRIX?

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ABSTRACT

It is shown how to single out a particular diagonal of a biinfinite band matrix A as its main diagonal, using the decomposition of the solution set of $A\underset{\sim}{x} = \underset{\sim}{0}$ into those which are bounded at ∞ and those which are bounded at $-\infty$. As an application, it is proved that the inverse of the coefficient matrix for the system satisfied by the B-splines coefficients of the cubic spline interpolant at knots is checkerboard and that, under certain assumptions, the local mesh ratio must be bounded.

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SIGNIFICANCE AND EXPLANATION

Spline approximation is often most effective when the breakpoint (knot) sequence can be chosen suitably nonuniform. At the same time, standard spline approximation schemes (such as least-squares approximation by splines) are so far only known to be bounded as long as the breakpoint sequence is almost uniform. Any such bound is obtained (explicitly or implicitly) in terms of a bound on the inverse of certain matrices which are banded. Any attempt at establishing bounds for more general breakpoint sequences must therefore come to grips with the inverses of these band matrices. The hope is the Demko's discovery of the exponential decay of band matrix inverses will lead eventually to those desired bounds.

In the present report, a specific question concerning the boundedness of cubic spline interpolation at breakpoints in terms of the local mesh ratio leads to a description of the inverse of a biinfinite band matrix A in terms of the behavior of the solutions of the homogeneous problem $A\underline{x} = \underline{0}$. This description should be of use in the analysis of other spline approximation schemes as well.

WHAT IS THE MAIN DIAGONAL OF A BIINFINITE BAND MATRIX?

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INTRODUCTION

The study of approximation by splines on a biinfinite knot sequence leads to linear systems

$$A\underline{x} = \underline{b}$$

with a banded biinfinite coefficient matrix A. Questions as to the existence of A^{-1}, its boundedness or its possible checkerboard nature need to be answered, and these, in turn, raise the question of which diagonal of A may be the main diagonal.

For example, it is well known that the inverse of a finite totally positive matrix A is checkerboard, i.e.,

$$(-)^{i+j}A^{-1}(i,j) \geq 0, \quad \text{all} \quad i,j$$

and, in particular, the entries of the main diagonal of A^{-1} are positive. One would expect the same statement to be true when A is biinfinite, but it is not clear a priori which diagonals of A^{-1} will be positive and which negative.

Again, in one approach toward proving that the inverse of a biinfinite totally positive matrix A is checkerboard, one would try to show that the inverse is approximated in some pointwise sense by the inverse of finite sections of A, whose checkerboard pattern is then known. Now, as we will show

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below, if there is any convergence at all, then these finite sections are necessarily principal with respect to one fixed diagonal of A. That diagonal then has earned the epithet "main".

Here is an outline. In Section 1, even-order spline interpolation at knots is discussed, since I was led to wonder about the main diagonals of biinfinite matrices because of an argument with C.A. Micchelli concerning mesh ratio restrictions for that scheme. Section 2 contains a discussion of biinfinite band matrices, in particular some propositions regarding existence and character of their inverses. In the last section, some of these results are applied to cubic spline interpolation at a biinfinite knot sequence, giving me an opportunity to correct a mistake in [4].

1. MESH RATIO RESTRICTIONS IN EVEN-ORDER SPLINE INTERPOLATION AT KNOTS

Let $m\$_{k,\underline{t}}$ be the normed linear space of bounded splines of order k with knot sequence $\underline{t} = (t_i)$ and the sup-norm. We take the knot sequence $\underline{t}$ to be biinfinite and strictly increasing. Also, let $\underline{\tau}$ be a strictly increasing biinfinite sequence. The problem is to determine for given $\underline{a} \in \ell_\infty$ an f in $m\$_{k,\underline{t}}$ with $f(\tau_i) = a_i$, all i. I call this interpolation problem correct (others have called it "poised") if it has exactly one solution for every $\underline{a} \in \ell_\infty$, i.e., if

$$R : mS_{k,\underline{t}} \longrightarrow \ell_\infty : f \longmapsto f|_{\underline{\tau}} := (f(\tau_i))_{-\infty}^{\infty}$$

is invertible. Note that R^{-1} is necessarily bounded if it exists.

This interpolation problem has received particular attention in the special case of even-order spline interpolation at knots, i.e.,

$$\underline{\tau} = \underline{t} \quad \text{and} \quad k \text{ even}.$$

See, e.g., [6] for a recent survey. I showed in my talk [4] at the last Bonn conference that this interpolation problem is correct in case the global mesh ratio

$$M_{\underline{t}} := \sup_{i,j} \Delta t_i / \Delta t_j$$

is finite. I also stated there without proof that, for a correct problem, the local mesh ratio

$$m_{\underline{t}} := \sup_{|i-j|=1} \Delta t_i/\Delta t_j$$

would have to be finite, since it is possible to bound this mesh ratio in terms of $\|R^{-1}\|$. It was this claim which Micchelli doubted when we discussed various possible sufficient conditions for the correctness of the interpolation problem last summer. Now, my claim was based on the corresponding result in case of a finite knot sequence, in [3]. Here is an adaptation of the argument there to the present bi-infinite context.

Supposing the problem correct, write the interpolant $R^{-1}\underline{f}$ to the particular data sequence $\underline{f} := ((-)^i)$ in terms of the normalized B-splines of order k for $\underline{t}$,

$$R^{-1}\underline{f} = \sum_j \alpha_j N_{j,k}$$

Then, from [2],

$$D_k^{-1}\|\underline{\alpha}\|_\infty \leq \|R^{-1}\underline{f}\|_\infty \leq \|\underline{\alpha}\|_\infty$$

for some positive constant D_k independent of $\underline{t}$. Now, for any particular i ,

$$\frac{2(-)^{i+1}}{\Delta t_i} = [t_i,t_{i+1}]R^{-1}\underline{f} = (R^{-1}\underline{f})'(\xi) = \Sigma\frac{\alpha_j-\alpha_{j-1}}{t_{j+k-1}-t_j}N_{j,k-1}(\xi)$$

Since $(N_{j,k-1})$ forms a partition of unity and $N_{j,k-1}$ has its support in (t_j,t_{j+k-1}) while $\xi \in (t_i,t_{i+1})$, this implies that

$$\min_{i-k+2\leq j\leq i} \frac{t_{j+k-1}-t_j}{\Delta t_i} \leq D_k\|R^{-1}\|$$

and it was from this inequality that I had drawn a bound for $m_{\underline{t}}$ in terms of $\|R^{-1}\|$. But, actually this inequality is not strong enough for such a conclusion in case, e.g., $t_{j+k-1}-t_j$ is a decreasing function of j . The desired conclusion can be drawn, though, if we are certain that the α_j alternate, and alternate in the right way.

PROPOSITION 1. If the B-spline coefficients $\underline{\alpha}$ of the interpolant $R^{-1}\underline{f}$ to the data sequence $\underline{f} = ((-)^i)$ alternate in such a way that $(-)^j\alpha_j \geq 0$, all j, then $m_{\underline{t}} \leq D_k\|R^{-1}\|$.

Proof. Under this additional assumption, we have $(-)^{i+1}(\alpha_j-\alpha_{j-1}) \le 0$ for $j = i$ and also for $j = i-k+2$ (since k is even), hence from the above

$$\frac{2}{\Delta t_i} \le \sum_{j=i-k+3}^{i-1} (-)^{i+1}\frac{\alpha_j-\alpha_{j-1}}{t_{j+k-1}-t_j} N_{j,k-1}(\xi)$$

$$\le 2\|\underline{\alpha}\|_\infty/\min\{t_{j+k-1}-t_j : i-k+3 \le j \le i-1\}$$

and therefore

$$\frac{\Delta t_{i-1} + \Delta t_{i+1}}{\Delta t_i} \le \min_{i-k+3\le j\le i-1} \frac{t_{j+k-1}-t_j}{\Delta t_i} \le D_k\|R^{-1}\|$$

which does imply the desired result.

This leaves open the question as to when we can expect the alternation assumption to hold. In a finite-dimensional situation, the alternation is immediate because of the total positivity of the coefficient matrix of the linear system

$$\sum_j \alpha_j N_{j,k}(t_i) = (-)^i , \quad \text{all } i$$

But the sense of the alternation depends on just what the range of i and j here is. Specifically, if $i = I+1,\ldots,I+n$, $j = J+1, \ldots, J+n$, then (assuming that $J < I < J+k$)

$$(-)^{I+j}\, \alpha_{J+j} > 0 , \text{ all } j .$$

In the biinfinite case, it is not even clear that $\underline{\alpha}$ alternates. In order to investigate this question further, we now turn to an analysis of biinfinite band matrices.

2. THE INVERSE OF A BIINFINITE BAND MATRIX

In order to give our definition of main diagonal, we need notation to describe sections of biinfinite matrices and their relationship to the linear map represented by that matrix.

Let I, J be integer intervals. Then

$$A_{I,J} := A|_{I\times J} = (A(i,j))_{i\in I, j\in J}$$

denotes the corresponding section of the biinfinite matrix A. We can think of $A_{I,J}$ simply as a $|I|\times|J|$ matrix. But, $A_{I,J}$ also describes the nontrivial part of the linear map

$$P_I A P_J$$

with

$$(P_I \underline{a})(i) := \begin{cases} \underline{a}(i) & , i \in I \\ 0 & , i \notin I \end{cases}$$

More precisely, $A_{I,J}$ is the matrix representation (with respect to the canonical basis) of the linear map

$$P_I(A|_{\operatorname{ran} P_J})$$

and we will not distinguish between these two.

Here and below, we will use the alternative notation $\underline{a}(i)$ for the i-th entry of the sequence $\underline{a}$, which is consonant with the notation $A(i,j)$ for the (i,j)-entry of the matrix A.

DEFINITION. The bounded and boundedly invertible biinfinite matrix A (as a map on $\ell_\infty(\mathbb{Z})$, say) has its r-th diagonal as main diagonal := the matrices $(A_{I,I+r})^{-1}$ converge finitely to A^{-1} as $I \longrightarrow \mathbb{Z}$, i.e., $A_{I,I+r}$ is invertible for all sufficiently large finite intervals I and

$$A^{-1}(i,j) = \lim_{I \to \mathbb{Z}} (A_{I,I+r})^{-1}(i,j) , \quad \text{all } i,j .$$

Here, we have used the abbreviation

$$I+r := \{i+r : i \in I\} .$$

For example, suppose that both A and A^{-1} are upper triangular in the usual meaning of the word, i.e.,

$$A(i,j) = A^{-1}(i,j) = 0 \quad \text{for} \quad i > j .$$

Then

$$\delta_{ij} = \sum_{k=i}^{j} A(i,k)A^{-1}(k,j)$$

showing that $(A^{-1})_{I,I} = (A_{I,I})^{-1}$. In this case then, diagonal 0 is the main diagonal of A (as usual!). But now let E be the map or matrix of the left shift,

$$(E\underline{a})(i) := \underline{a}(i+1) , \text{ all } i ,$$

and let $r \in \mathbb{Z}$. Then $E^r A$ is also invertible, with inverse $A^{-1}E^{-r}$. But now

$$(E^r A)(i,j) = A(i+r,j) , \quad (E^r A)^{-1}(i,j) = A^{-1}(i,j+r)$$

hence now

$$((E^rA)_{I-r,I})^{-1} = (A_{I,I})^{-1} = A^{-1}_{I,I} = (E^rA)^{-1}_{I,I-r}$$

In other words, E^rA has diagonal r as its main diagonal (while $(E^rA)^{-1}$ has diagonal $-r$ as its main diagonal).

For matrices which are not triangular with triangular inverse, it is much more difficult to ascertain whether or not they even have a main diagonal, let alone which diagonal it might be. We now discuss this question in the context of banded matrices.

DEFINITION. We say that the biinfinite matrix A is m-banded if

(1) $A(i,j) = 0$ for $j \notin [i,i+m]$

(2) $A(i,i)A(i,i+m) \neq 0$, all i.

Thus an m-banded matrix has at most $m+1$ nonzero bands. If A is m-banded, then, for any r, E^rA also has just $m+1$ possibly nonzero contiguous bands and so could, with justification, also be called m-banded. But we will use the term "m-banded" only as described in order to suppress an additional inessential parameter. The other assumption, viz. the nonvanishing of the first and last band, is a nontrivial one. It makes certain statements simpler and is satisfied in the case of spline interpolation at knots.

An m-banded matrix A gives rise to a linear map on $\mathbb{R}^{\mathbb{Z}}$ which we will identify with A. This map has an m-dimensional nullspace or kernel,

$$\mathfrak{N} := \mathfrak{N}_A := \{\underline{f} \in \mathbb{R}^{\mathbb{Z}} : A\underline{f} = 0\}$$

In particular,

(3) for every i, $\mathfrak{N} \longrightarrow \mathbb{R}^m : \underline{f} \longmapsto \underline{f}\big|_{[i+1,i+m]}$ is 1-1 and onto

because of (2). Conversely, if $\mathfrak{N}$ is an m-dimensional subspace of $\mathbb{R}^{\mathbb{Z}}$ for which (3) holds, then there is, up to left multiplication by a diagonal matrix, exactly one m-banded matrix having $\mathfrak{N}$ as its nullspace.

We also introduce two subspaces of $\mathfrak{N}$,

$$\mathfrak{N}^+ := \{\underline{f} \in \mathfrak{N} : \overline{\lim_{i\to\infty}}\, \underline{f}(i) < \infty\}, \quad m^+ := \dim \mathfrak{N}^+$$

$$\mathfrak{N}^- := \{\underline{f} \in \mathfrak{N} : \overline{\lim_{i\to-\infty}}\, \underline{f}(i) < \infty\}, \quad m^- := \dim \mathfrak{N}^-$$

From now on, we assume that A is m-banded and bounded on ℓ_∞. Then the i-th column $A^{-1}(\cdot,i)$ of its inverse, if it exists, would solve the linear system

$$A\underline{x} = (\delta_{ij}) .$$

The following proposition is therefore a first step toward understanding the inverse of an m-banded matrix.

PROPOSITION 2. For all i, there exists exactly one $\underline{L}_i \in \ell_\infty$ such that $A\underline{L}_i = (\delta_{ij})$ if and only if $\mathfrak{N} = \mathfrak{N}^- \oplus \mathfrak{N}^+$.

Proof. Since $\mathfrak{N}^- \cap \mathfrak{N}^+$ is the kernel of $A|_{\ell_\infty}$, there is at most one solution (for any particular i) if and only if $\mathfrak{N}^- \cap \mathfrak{N}^+ = \{0\}$. Hence it is sufficient to prove that, given uniqueness, there is a solution for every i iff $m^- + m^+ = m$.

For this, note that $\underline{L}_i \in \ell_\infty$ satisfies $A\underline{L}_i = (\delta_{ij})$ iff

(4) $$\underline{L}_i(j) = \begin{cases} \underline{L}_i^-(j) , & j<i+m \\ \underline{L}_i^+(j) , & j>i \end{cases} , \text{ with } \begin{matrix} \underline{L}_i^- \in \mathfrak{N}^- \\ \underline{L}_i^+ \in \mathfrak{N}^+ \end{matrix}$$

and

(5) $$\sum_{j=i}^{i+m} A(i,j)\underline{L}_i(j) = 1 .$$

Here, $\underline{L}_i^-$, $\underline{L}_i^+$ is the extension of $\underline{L}_i|_{[i,i+m-1]}$ and $\underline{L}_i|_{[i+1,i+m]}$, respectively, to an element of $\mathfrak{N}$.

Now to see that $m^- + m^+ = m$ implies existence of $\underline{L}_i$, note that (4) and (5) constitute a linear system

(6) $$B_i(\underline{L}_i|_{[i,i+m]}) = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

in the $m+1$ unknowns $\underline{L}_i(i), \ldots, \underline{L}_i(i+m)$, with the first $m-m^-$ homogeneous conditions ensuring that the extension $\underline{L}_i^-$ of $\underline{L}_i|_{[i,i+m-1]}$ to an element of $\mathfrak{N}$ lies in $\mathfrak{N}^-$, and the next $m-m^+$ homogeneous conditions ensuring that the extension $\underline{L}_i^+$ of $\underline{L}_i|_{[i+1,i+m]}$ to an element of $\mathfrak{N}$ lies in $\mathfrak{N}^+$, and the last, the only inhomogeneous, condition being (5). But if now $m^- + m^+ = m$, then (6) has as many equations as unknowns and, as we already know that it has at most one solution, the existence of a solution follows.

Conversely, assuming the existence of a solution for every i, consider the maps

$$\phi^* : \mathbb{R}^{\overline{m}+1} \longrightarrow \mathfrak{N} : \underline{a} \longmapsto \sum_{j=0}^{\overline{m}} a_j \underline{L}^*_{i+j}$$

with $*$ standing for $+$ or $-$, and $\overline{m} := m^- + m^+$. Then

$$\dim \ker \phi^* = \overline{m}+1 - \dim \operatorname{ran} \phi^* \geq \overline{m}+1-m^* = \begin{cases} m^-+1 , & * = + \\ m^+ +1 , & * = - \end{cases}$$

Consequently, there exists $\underline{a} \in \ker \phi^+ \cap \ker \phi^- \setminus \{0\}$. For this $\underline{a}$,

$$\underline{M} := \sum_{j=0}^{\overline{m}} a_j \underline{L}_{i+j} \neq 0$$

since $(\underline{L}_j)$ is obviously linearly independent. On the other hand, since

$$\underline{L}_{i+j}(s) = \begin{cases} \underline{L}^-_{i+j}(s) , & \text{for } s < i+m \\ \underline{L}^+_{i+j}(s) , & \text{for } s > i+\overline{m} \end{cases} , \quad j=0,\ldots,\overline{m} ,$$

we find

$$\underline{M} = \begin{cases} \Sigma_0^{\overline{m}} a_j \underline{L}^-_{i+j} & \text{on }]-\infty, i+m[\\ \Sigma_0^{\overline{m}} a_j \underline{L}^+_{i+j} & \text{on }]i+\overline{m}, \infty[\end{cases}$$

and therefore, by choice of $\underline{a}$, $\underline{M}(s) = 0$ for $s < i+m$ and $s > i+\overline{m}$. This implies $\overline{m} \geq m$, and therefore, since by assumption $\mathfrak{N}^+ \cap \mathfrak{N}^- = \{0\}$, i.e., $m^+ + m^- \leq m$, the conclusion $m^+ + m^- = m$ follows.

Next, we characterize bounded invertibility of a bounded m-banded matrix A in terms of $\mathfrak{N}$.

PROPOSITION 3. A is boundedly invertible if and only if (i) $\mathfrak{N} = \mathfrak{N}^- \oplus \mathfrak{N}^+$, (ii) the elements of $\mathfrak{N}^-$ and $\mathfrak{N}^+$ decay exponentially and, (iii) for each i, the matrix B_i in (6) can be chosen so that $\sup_i \|B_i^{-1}\| < \infty$.

Concerning the exponential decay, I had proved at the last Bonn conference that, for a bounded and boundedly invertible m-banded matrix A on some ℓ_p with $p < \infty$, there exists const so that, for all $\underline{f} \in \mathfrak{N}^-$ and all i,

$$\|\underline{f}^{(r)}\|_p^p \geq \text{const } \Lambda^r \|\underline{f}^{(0)}\|_p^p , \quad r=1,2,3,\ldots$$

with

$$\Lambda := \left|\frac{1+\kappa^p}{1-\kappa^p}\right| , \quad \kappa := \|A\|\,\|A^{-1}\|$$

and

$$\underline{f}^{(r)} := \underline{f}|_{[i+rn,i+(r+1)m-1]} ,$$

and an analogous statement for $\underline{f} \in \mathfrak{N}^+$. The exponential decay mentioned in condition (ii) of the proposition is meant in this sense.

The proof of the necessity of the exponential decay rests on Demko's [5] nice idea. As a footnote, I would like to record here that, in response to my talk, S.Demko, at the end of his talk at the present conference, made the point that he had been materially helped by a referee's report authored, as it happens, by T. Lucas.

Finally, the proof of the necessity of the last condition is a bit tricky.

On the other hand, the sufficiency of the three conditions is immediate since they insure that the $\underline{L}_i$'s , constructed in Proposition 2 on the strength of (i), are ℓ_∞-bounded uniformly in i and decay exponentially, hence the matrix $[\ldots,\underline{L}_i,\ldots]$ is a bounded map on ℓ_p for every p , etc.

The complete proof of Proposition 3 (and of Proposition 4 to follow) can be found in [7]. Finally, we state a necessary and sufficient condition for such a bounded m-banded matrix A to have a main diagonal.

PROPOSITION 4. A <u>has diagonal</u> r <u>as its main diagonal if and only if (i)</u> A <u>is boundedly invertible, (ii)</u> $r = m^+$<u>, (iii) there exists a positive</u> const <u>so that, for all large</u> n ,

$$\underline{\text{for all}}\ \underline{f} \in \mathfrak{N}^+ ,\quad \|\underline{f}|_{[-n,-n+m^+-1]}\| \geq \text{const}\ \|\underline{f}|_{[-n,-n+m]}\|$$

<u>and</u>

$$\underline{\text{for all}}\ \underline{f} \in \mathfrak{N}^- ,\quad \|\underline{f}|_{[n+m^++1,n+m]}\| \geq \text{const}\ \|\underline{f}|_{[n,n+m]}\| .$$

In effect, under these assumptions, we can construct the column $\underline{L}_i^{(I)}$ of $(A_{I,I+r})^{-1}$ for all large I as a modification of the corresponding column $\underline{L}_i$ of A^{-1} ,

$$\underline{L}_i^{(I)} = \underline{L}_i - \underline{L}^+ - \underline{L}^- \quad \text{on}\ I+r$$

with

$$\underline{L}^+ + \underline{L}^- = \underline{L}_i \quad \text{on} \quad I \setminus (I+r) \cup (I+m) \setminus (I+r)$$

and $\underline{L}^* \in \mathfrak{N}^*$. This guarantees that $\|\underline{L}_{*i}^{(I)}\| \sim \|\underline{L}_i\|$ while, because of the exponential decay of $\underline{L}^*$, $\underline{L}_i^{(I)} \sim \underline{L}_i$ away from the boundary of $I+r$.

3. CUBIC SPLINE INTERPOLATION AT KNOTS

In this section, we establish the checkerboard pattern for A^{-1} in case

$$(7) \quad A = (N_{j,4}(t_i))$$

of cubic spline interpolation at knots.

Given that the interpolation problem is correct, we now know that there are just three possibilities: $m^+ = 0,1,2$.

Case $m^+=0$. In this case, $\mathfrak{N} = \mathfrak{N}^-$, and from (4) we see that A^{-1} is upper triangular. Therefore, as discussed in Section 2, the first diagonal of A is main, i.e., diagonal 1 in our way (7) of writing A. It follows that

$$(-)^{i+j+1} A^{-1}(i,j) \geq 0, \quad \text{all } i,j$$

and, in particular, the solution $\underline{\alpha}$ of $A\underline{\alpha} = ((-)^i)$ satisfies $(-)^{i+1}\alpha_i > 0$, all i. Thus, $\underline{\alpha}$ alternates but in the wrong sense if we are after bounding the local mesh ratio in terms of $\|R^{-1}\|$ using the argument of Proposition 1. In fact, it is not difficult to construct a knot sequence $\underline{t}$ for which the interpolation problem is correct and for which $t_i \longrightarrow 0$ as $i \longrightarrow -\infty$ so strongly that $m_{\underline{t}} = \infty$. My statement at the last Bonn conference ([4;p.48]) that $m_{\underline{t}} \leq \text{const} \|R^{-1}\|$ must therefore be qualified to exclude the case $m^+ = 0$ and the analogous case $m^+ = 2$.

Case $m^+=1$. We find it convenient to associate with the element $\underline{L} \in \mathfrak{N}$ the nullspline

$$L := \sum_j \underline{L}(j)\, N_{j,4}$$

for which it supplies the B-spline coefficients. We claim that

$$(8) \quad \underline{L} \in \mathfrak{N}^- \setminus \{0\} \text{ implies } L'(t_i)L''(t_i) > 0, \text{ all } i.$$

Indeed, $\mathfrak{N}$ contains a sequence $(\underline{L}_{[i]})$ so that $L'_{[i]}(t_i)L''_{[i]}(t_i) > 0$, hence $L'_{[i]}(t_j)L''_{[i]}(t_j) > 0$ for $j \geq i$, by [1]. Since $\mathfrak{N}$ is finite-dimensional, a properly normalized subsequence then has a limit L for which $L'(t_i)L''(t_i) \geq 0$, all i. But then, by [1],

$$L'(t_i)L''(t_i) > 0 \text{, all } i$$
$$\|L\|_{i+j} \geq 2^j\|L\|_i \text{, all } j \geq i$$

with

$$\|L\|_j := \max\{|L'(t_j)|, |L''(t_j)|\} .$$

Suppose now that, by way of contradiction, $\underline{M} \in \mathfrak{N}^- \setminus \{0\}$ satisfies $M'(t_i)M''(t_i) \leq 0$ for some i. Then, again by [1], $M'(t_j)M''(t_j) < 0$ for $j < i$ and

$$\|M\|_{i-j} \geq 2^j\|M\|_i \text{, } j=1,2,3,\ldots,$$

showing that then $M'(t_j)$ and $M''(t_j)$ both would increase exponentially as $j \longrightarrow -\infty$, while M itself stays bounded since $\underline{M} \in \mathfrak{N}^-$. This would imply that Δt_j decreases exponentially as $j \longrightarrow -\infty$ and then, given that $2^{-j}\|L\|_i \geq \|L\|_{i-j}$ for $j=1,2,\ldots,$ $\underline{L}$ would surely also be in $\mathfrak{N}^-$ contradicting the fact that $m^- = 1$.

One proves analogously that

(9) $\underline{L} \in \mathfrak{N}^+ \setminus \{0\}$ implies $L'(t_i)L''(t_i) < 0$, all i.

(8) and (9) imply (see [1]) that

(10) for all $\underline{L} \in \mathfrak{N}^* \setminus \{0\}$, $L''(t_i)L''(t_{i+1}) < 0$, all i.

Next, we claim that

(11) for all $\underline{L} \in \mathfrak{N}^* \setminus \{0\}$, $\underline{L}_j\, L''(t_{j+2}) < 0$, all j.

For this, recall (e.g., from[2;p.270]) that $f = \Sigma\alpha_j N_{j,4}$ implies

(12) $$\alpha_j = \sum_{r<4} (-)^{3-r}\psi^{(3-r)}(\tau)f^{(r)}(\tau)$$

for any $\tau \in (t_j, t_{j+4})$, with

$$\psi(x) := (t_{j+1}-x)(t_{j+2}-x)(t_{j+3}-x)/3! .$$

Since $L = \Sigma\, \underline{L}_j\, N_{j,4}$, we then get

$$\underline{L}_j = \psi''(t_{j+1})L'(t_{j+1}) - \psi'(t_{j+1})L''(t_{j+1})$$

while $\psi''(t_{j+1}) > 0 > \psi'(t_{j+1})$. Therefore, with (8),

$$\underline{L}_j L''(t_{j+1}) = \psi''(t_{j+1})L'(t_{j+1})L''(t_{j+1}) - \psi'(t_{j+1})[L''(t_{j+1})]^2 > 0$$

if $\underline{L} \in \mathfrak{N}^- \setminus \{0\}$, and (10) now finishes the proof of (11) for this case. The case $\underline{L} \in \mathfrak{N}^+ \setminus \{0\}$ uses (12) with $\tau = t_{j+3}$ instead.

Finally, let now $\underline{L}$ be one of the columns of A^{-1},

$$\underline{L} = A^{-1}(\cdot,i)$$

say. Since $m^- = m^+ = 1$, we have

$$(13) \quad \underline{L}(j) = \begin{cases} \underline{L}^-(j), & \text{for } j < i+2 \\ \underline{L}^+(j), & \text{for } j > i \end{cases}$$

with neither $\underline{L}^-$ nor $\underline{L}^+$ just 0 . Consequently, by (10) and (11), $\underline{L}$ alternates (strictly) and, as to the sense of that alternation, we have from (12) (with $j = i-2$) that

$$\begin{aligned} A^{-1}(i-2,i) &= \psi''(t_{i-1})L'(t_{i-1}) - \psi'(t_{i-1})L''(t_{i-1}) \\ &= \psi''(t_{i+1})L'(t_{i+1}) - \psi'(t_{i+1})L''(t_{i+1}) \end{aligned}$$

while, from (13), (8) and (9),

$$\psi''(t_{i-1}) > 0 > \psi'(t_{i-1}) \ , \quad L'(t_{i-1})L''(t_{i-1}) > 0$$

$$\psi''(t_{i+1}),\ \psi'(t_{i+1}) < 0 \ , \quad L'(t_{i+1})L''(t_{i+1}) < 0$$

We conclude that

$$\text{sign } A^{-1}(i-2,i) = \text{sign } L''(t_{i-1}) = \text{sign } L''(t_{i+1}) .$$

But this implies that $A^{-1}(i-2,i) > 0$, since the contrary assumption would give $L'(t_i^-) > 0 > L'(t_i^+)$ (since $L(t_i) = 1$), an impossibility.

In conclusion, if $m^- = m^+ = 1$, then

$$(14) \quad (-)^{i+j}A^{-1}(i,j) > 0 \ , \text{ all } i,j,$$

and, in particular, the solution of the linear system $A\underline{\alpha} = ((-)^i)$ does satisfy the condition $(-)^i\alpha_i > 0$, all i, needed in the argument for the finiteness of the local mesh ratio.

Note that we proved (14) here without recourse to finite sections. Even in this simple case, I still do not know whether the matrix A has a main diagonal (though I don't think it would be very hard to prove).

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It is shown how to single out a particular diagonal of a biinfinite band matrix A as its main diagonal, using the decomposition of the solution set of $A\underline{x} = \underline{0}$ into those which are bounded at ∞ and those which are bounded at $-\infty$. As an application, it is proved that the inverse of the coefficient matrix for the system satisfied by the B-splines coefficients of the cubic spline interpolant at knots is checkerboard and that, under certain assumptions, the local mesh ratio must be bounded.

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